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A monomial cycle basis on Koszul homology modules

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Abstract

We give a class of p-Borel principal ideals of a polynomial algebra over a field K for which the graded Betti numbers do not depend on the characteristic of K and the Koszul homology modules have a monomial cyclic basis.

MSC: 13D02; 13D07; 13P10; 13D99

0. Introduction

Let K be an infinite field, $S = K[x_1, \dots, x_n]$, $n \geq 2$, the polynomial ring over K and $I \subset S$ a graded ideal. Let $\leq$ be a term order on the monomials of S satisfying $x_1 > x_2 > \cdots > x_n$ and $\mathrm{Gin}(I)$ the generic initial ideal of I with respect to $\leq$. $\mathrm{Gin}(I)$ is a Borel fixed ideal, that is, it is fixed by the Borel group of invertible upper triangular matrices. Let M be a graded S-module and $\beta_{ij}(M) = \beta_{ij}$ the graded Betti numbers of M. The Castelnuovo–Mumford regularity of M is $\mathrm{reg}(M) = \max\{j - i : \beta_{ij}(M) \neq 0\}$. If $\mathrm{char}\, K = 0$ then $\mathrm{Gin}(I)$ is strongly stable, that is, it is monomial and for each monomial u of $\mathrm{Gin}(I)$ and $1 \leq j < i \leq n$ such that $x_i | u$ it follows $x_j(u/x_i) \in \mathrm{Gin}(I)$. Then $\mathrm{reg}(\mathrm{Gin}(I))$ is the highest degree of minimal generators of $\mathrm{Gin}(I)$ by Eliahou and Kervaire [5]. If $\mathrm{char}\, K = p > 0$ then Borel fixed ideals are just the so called p-Borel ideals and they are not necessarily strongly stable, and it is hard to give a formula for the regularity of these ideals. Let I be a monomial ideal of S, u a monomial of I and $\nu_i(u)$ be the highest power of x_i dividing u. Let a, b be two integers and $a = \sum_{i \geq 0} a_i p^i$, $b = \sum_{i \geq 0} b_i p^i$ be the p-adic expansions of a, b, respectively. We say that $a \leq_p b$ if $a_i \leq b_i$ for all i. It is well known that a monomial ideal I is p-Borel if for any monomial $u \in I$ and $1 \leq j < i \leq n$ and a positive integer t such that $t \leq_p \nu_i(u)$ it holds that $x_j^t(u/x_i^t) \in I$. This is a pure combinatorial description of the p-Borel ideals which can be given independently of the characteristic of K. Let u be a monomial of S and $J = \langle u \rangle$ the smallest p-Borel ideal containing u. J is called the principal p-Borel ideal. For such ideals there exists a complicated formula for regularity in terms of u conjectured by Pardue [8] and proved in two papers [2,6] (another proof is given in [7]).

The principal p-Borel ideals $I \subset S$ such that S/I is Cohen–Macaulay have the form $I = \prod_{j=0}^{s} (m^{[p^j]})^{\alpha_j}$, $0 \leq \alpha_j < p$, where $m^{[p^j]} = (x_1^{p^j}, \dots, x_n^{p^j})$. For these ideals the description of a canonical monomial cycle

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basis of the Koszul homology module $H_i(x; S/I)$ given by Aramova and Herzog [2] is well known (see details in Theorem 2.1). One can easily see from this description that $\beta_{ij}(S/I)$ does not depend on the characteristic of the field K for all i, j.

Now let I be the p-Borel ideal generated by the monomial $x_{n-1}^{\gamma}x_n^{\alpha}$ for some integer $\gamma, \alpha \geq 0$, that is

$$I = \prod_{j=0}^{s}((m_{n-1}^{[p^j]})^{\gamma_j}(m^{[p^j]})^{\alpha_j}),$$

where $m_{n-1} = (x_1, \ldots, x_{n-1})$, and γ_j, α_j are defined by the p-adic expansions of γ, α, respectively. Suppose that $\alpha_j + \gamma_j < p$ for all $0 \leq j \leq s$. Then $H_i(x; S/I)$ has a monomial cycle basis for all $i \geq 2$, and $\beta_{ij}(S/I)$ does not depend on the characteristic of K for all i, j (see Theorem 2.7).

We saw that in some cases of principal p-Borel ideals there exists a monomial cycle basis for the homology modules of S/I. What is the general case? If $I \subset S$ is a monomial ideal then $H_2(x; S/I)$ has a monomial cycle basis (see Theorem 1.5). Unfortunately, in general there are no monomial cycle bases even on the Koszul homology modules of principal p-Borel ideals as shown by our Example 1.6. However if I is a principal p-Borel ideal then $H_3(x; S/I)$ has a binomial cycle basis (see [10]), but we have no idea whether $\beta_{ij}(S/I)$ depends or does not depend on the characteristic of K, except for the cases given by Theorems 2.1 and 2.7. Note that the graded Betti numbers of the monomial ideal I_Δ associated with the simplicial complex Δ defined by the triangulation of the real projective plane depend on the characteristic of K (see the end of [4, Section 5.3]). We express our thanks to J. Herzog in particular for some discussions concerning Theorem 2.7.

1. Cycles of Koszul homology modules of monomial ideals

Let $S = K[x_1, \ldots, x_n]$ be a polynomial algebra over a field K and $I \subset S$ a monomial ideal. An element $z \in K_i(x; S/I)$ has the form $z = \sum_{j=1}^{s}\gamma_j u_j e_{\sigma_j}$, $\gamma_j \in K^*$, u_j monomials, $\sigma_j \subset \{1, \ldots n\}$, $|\sigma_j| = i$ for all $1 \leq j \leq s$ (if $\tau = \{r_1, \ldots, r_i\}$ for some $1 \leq r_1 < \cdots < r_i \leq n$ then e_τ denotes $e_{r_1} \wedge \cdots \wedge e_{r_i}$). Since I is a monomial the Koszul antiderivation ∂ is multigraded and each cycle of $K_i(x; S/I)$ is a sum of multigraded cycles. A cycle z of the above form is *multigraded* if $u_j x_{\sigma_j} = u_1 x_{\sigma_1}$ for all $1 \leq j \leq s$; here $x_{\sigma_1} = \prod_{k\in\sigma_1} x_k$. We define $m(u_j) = \max\{i; x_i \mid u_j\}$ and $m(\sigma_j) = m(x_{\sigma_j})$. Note that we may suppose that in $z\sigma_j \neq \sigma_t$ for $j \neq t$ because otherwise it follows that $u_j = u_t$ (z is multigraded) and so we may reduce the sum. The element $u_j e_{\sigma_j}$ is a monomial cycle if $\partial(u_j e_{\sigma_j}) = 0$, that is $x_t u_j \in I$ for all $t \in \sigma_j$.

We introduce a total order on the monomial elements ue_σ of $K_i(x; S/I)$ (u monomial) as follows: "$ue_\sigma \geq ve_\tau$" if either "$x_\sigma \geq_{\text{rlex}} x_\tau$" or "$x_\sigma = x_\tau$" and "$u >_{\text{rlex}} v$"; here rlex denotes the reverse lexicographical order on the monomials of S. As usual, we define $\text{in}(z) = u_1 e_{\sigma_1}$ if $u_1 e_{\sigma_1} > u_j e_{\sigma_j}$ for all $j > 1$. A σ_j is called a *neighbour* in z of σ_1 if $|\sigma_j \setminus \sigma_1| = 1$.

Lemma 1.1. *If z is a cycle and σ_1 has no neighbour in z then $u_1 e_{\sigma_1}$ is a monomial cycle.*

Proof. Since z is a cycle all the terms of $\partial(u_1 e_{\sigma_1})$ should be reduced with terms of some $\partial(u_j e_{\sigma_j})$, $j > 1$. But this is possible only if σ_j is a neighbour of σ_1. $\square$

Lemma 1.2. *Let $z = \sum_{j=1}^{s}\gamma_j u_j e_{\sigma_j}$ be a multigraded cycle, $\gamma_j \in K^*$, u_j monomials, $\sigma_j \subset \{1, \ldots n\}$, $|\sigma_j| = i$ for all $1 \leq j \leq s$. Then the following statements hold:*

(1) *If $\text{in}(z) = u_1 e_{\sigma_1}$ and $m(u_1) > m(\sigma_1)$ then there exists a multigraded element $w \in B_i(x; S/I)$ such that $\text{in}(w) = \text{in}(z)$.*
(2) *For every multigraded cycle w there exists a multigraded cycle z of the above form in the same multigraded homology class as w such that $m(u_j) \leq m(\sigma_j)$ for all $1 \leq j \leq s$.*
(3) *If z is in the form given by (2) it follows that $m(\sigma_j) = m(\sigma_1)$ for all $1 \leq j \leq s$.*

Proof. (1) Take a $q > m(\sigma_1)$ such that $x_q \mid u_1$ and set $y = (u_1/x_q)e_{\sigma_1\cup\{q\}}$. We have that ∂y is the sum of $(u_1/x_q)(\partial e_{\sigma_1}) \wedge e_{\{q\}}$ with $+$ or $-$ $u_1 e_{\sigma_1}$. Thus $\text{in}(\partial y) = \text{in}(z)$.

(2) + (3) Subtracting from z such elements w of $B_i(x; S/I)$ we may arrive at the case $m(u_j) \leq m(\sigma_j)$. Since z is multigraded we get then $m(\sigma_j) = m(\sigma_1)$. $\square$

Lemma 1.3. *Let $z = \sum_{j=1}^{s} \gamma_j u_j e_{\sigma_j}$ be a multigraded cycle as in the above lemma. Suppose that $m(u_j) \le m(\sigma_j)$ for all $1 \le j \le s$. Then $x_r u_j \in I$ for $r = m(\sigma_1)$ and for all $1 \le j \le s$.*

Proof. By Lemma 1.2(3) we get $r = m(\sigma_1) = m(\sigma_j)$. The terms $x_r u_j e_{\sigma_j \setminus \{r\}}$ of $\partial(u_j e_{\sigma_j})$ cannot be reduced since the $\sigma_j \setminus \{r\}$ are all different. It follows necessarily that $x_r u_j \in I$ since z is a cycle. □

Let $\mathcal{M}_i(x; S/I)$ be the subspace of $K_i(x; S/I)$ generated by all monomial cycles.

Lemma 1.4. *Let $z = \sum_{j=1}^{s} \gamma_j u_j e_{\sigma_j}$ be a multigraded 2-cycle, $\gamma_j \in K^*$, u_j monomials, $\sigma_j \subset \{1, \ldots, n\}$, $|\sigma_j| = i$ for all $1 \le j \le s$. Suppose that $m(u_j) \le m(\sigma_j)$ for all j, $s > 1$ and $\mathrm{in}(z) = u_1 e_{\sigma_1}$. Then one of the following conditions holds:*

(1) $\mathrm{in}(z)$ *is a monomial cycle,*
(2) $\mathrm{in}(z) \equiv u_j e_{\sigma_j} \mathrm{mod}(B_2(x; S/I) + \mathcal{M}_2(x; S/I))$ *for some* $1 < j \le s$.

Proof. By Lemma 1.2(3) and our hypothesis we get $r = m(\sigma_1) = m(\sigma_j)$ for all $1 \le j \le s$. Let $\sigma_1 = \{a, r\}$. If $x_a u_1 \in I$ then $u_1 e_{\sigma_1}$ is a monomial cycle, by the above lemma. Otherwise, by Lemma 1.1 there exists a neighbour $\sigma_j = (\sigma_1 \setminus \{a\}) \cup \{b\}$ for some $1 \le b < r, a \ne b$. As $\mathrm{in}(z) = u_1 e_{\sigma_1}$ we have $\sigma_1 > \sigma_j$ and so $a < b$. From $x_a u_1 = x_b u_j$ (z is multigraded!) it follows that $x_b \mid u_1$. Set $y = (u_1/x_b) e_{\{a,b,r\}} \in K_3(x; S/I)$. We have

$$\partial y = -u_1 e_{\sigma_1} + u_j e_{\sigma_j} + (x_r u_1/x_b) e_{\{a,b\}}.$$

Using Lemma 1.3 we have $x_r u_t \in I$ for all $1 \le t \le s$. This shows that the last term of ∂y is a monomial cycle, which is enough. □

Theorem 1.5. *Every 2-cycle of $K_2(x; S/I)$ belongs to $B_2(x; S/I) + \mathcal{M}_2(x; S/I)$, that is coincides modulo $B_2(x; S/I)$ with a sum of monomial cycles. In particular, $H_2(x; S/I)$ has a monomial cycle basis.*

Proof. Note that given a 2-cycle z in the form of Lemma 1.2(2) $\mathrm{in}(z)$ can be substituted in z modulo $B_2(x; S/I)$ with one monomial term smaller than $\mathrm{in}(z)$ and some monomial cycles which can be removed from z (see Lemma 1.4). By recurrence we arrive finally at the case where z has just one term which must be then a monomial cycle. □

Unfortunately, in general there are not monomial cycle bases even on the Koszul homology modules of principal p-Borel ideals as shown by the following example:

Example 1.6. Let $n = 4$, that is $S = K[x_1, \ldots, x_4]$, and let I be the p-Borel ideal generated by the monomial $\{x_2 x_4^p\}$, that is $I = (x_1, x_2)(x_1^p, \ldots, x_4^p)$. Consider the element $z = x_2 x_3^{p-1} x_4^{p-1} e_{134} - x_1 x_3^{p-1} x_4^{p-1} e_{234} \in K_3(x; S/I)$. We see that z is a binomial cycle but $\mathrm{in}(z) = x_2 x_3^{p-1} x_4^{p-1} e_{134}$ is not a monomial cycle because $x_1 x_2 x_3^{p-1} x_4^{p-1} \notin I$. Note that z is multigraded and in its multigraded homology class another element has the form $z + \partial y$, $y \in K_4(x; S/I)$. Since y must be multigraded from the same multigraded class with z we see that the only possibility is to take $y = x_3^{p-1} x_4^{p-1} e_{1234}$. But then $z + \partial y = x_3^p x_4^{p-1} e_{124} - x_3^{p-1} x_4^p e_{123}$, which is also a binomial cycle, but not a sum of monomial cycles. It follows that there exist no monomial cycle in the homology class of z.

2. Monomial cycle basis on Koszul homology modules of some principal p-Borel ideals

The principal p-Borel ideals $I \subset S$ such that S/I is Cohen–Macaulay have the form $I = \prod_{j=0}^{s} (m^{[p^j]})^{\alpha_j}$, $0 \le \alpha_j < p$. For these ideals the description of a canonical monomial cycle basis of $H_i(x; S/I)$ is well known. Fix $2 \le i \le n$. Let $0 \le t \le s$ be an integer and for $v \in G(m^{\alpha_t})$ define $v' = v/x_{m(v)}$. Let $B_{it}(I)$ be the following set of elements from $K_i(x; S/I)$:

$$\left\{ w v'^{p^t} x_\sigma^{p^t - 1} e_\sigma : w \in G\left(\prod_{j>t} (m^{[p^j]})^{\alpha_j}\right), v \in G(m^{\alpha_t}), \sigma \subset \bar{n}, |\sigma| = i, m(\sigma) = m(v) \right\}$$

and $B_i(I) = \cup_{t=0}^{s} B_{it}(I)$, where $\bar{n} = \{1, \ldots, n\}$.

Theorem 2.1 (*Aramova–Herzog [2]*). *The elements of $B_i(I)$ are cycles in $K_i(x; S/I)$ and their homology classes form a basis in $H_i(x; S/I)$ for $i \geq 2$.*

Remark 2.2. This result holds independently of the characteristic of K, as we had pointed out, the definition of p-Borel ideals is pure combinatorial. But note that Theorem 2.1 does not hold if $\alpha_j \geq p$ for some j. Indeed, the ideal $I = (x_1, x_2)^4 \subset S = K[x_1, x_2]$ is strongly stable and a monomial basis of $H_2(x; S/I)$ is given by $T = \{x_1^3 e_1 \wedge e_2, x_1^2 x_2 e_1 \wedge e_2, x_1 x_2^2 e_1 \wedge e_2, x_2^3 e_1 \wedge e_2\}$ by [1] (see also [5]). Since $I = (x_1, x_2)^2(x_1^2, x_2^2)$ one can compute $B_0(I) = T$ and $B_1(I) = \{x_1 x_2 e_1 \wedge e_2\}$ but $x_1 x_2 e_1 \wedge e_2$ is not a cycle in $K_2(x; S/I)$. So the condition $\alpha_j < p$ is necessary and this is an obstruction for an extension of Theorem 2.1.

The question that appeared in Remark 2.2 perhaps can be solved by extending somehow Theorem 2.1 for the case where the α_j are arbitrary. In some special cases a possible tool could be the following lemma.

Lemma 2.3. *Let $I = \prod_{j=0}^{s} (m^{[p^j]})^{\alpha_j}$, where $\alpha_j \geq 0$ are arbitrary integers. If $n = 2$ then there exist some integers $0 \leq j_0 < j_1 < \cdots < j_k$ and some positive integers $(\gamma_t)_{0 \leq t \leq k}$ such that $\gamma_t < p^{j_{t+1} - j_t}$ for $t < k$ and $I = \prod_{t=0}^{k} (m^{[p^{j_t}]})^{\gamma_t}$.*

For the proof, apply by recurrence the relation $m^{p^t} m^{[p^t]} = (m^{p^t})^2$.

Set $I = \prod_{t=0}^{k} (m^{[p^{j_t}]})^{\gamma_t}$ as above but for any n and let $C_{it}(I)$ be the following set of elements from $K_i(x; S/I)$:

$$\{w v'^{p^{j_t}} x_\sigma^{p^{j_t}-1} e_\sigma : w \in G(\prod_{r>t} (m^{[p^{j_r}]})^{\gamma_r}), v \in G(m^{\gamma_t}), \sigma \subset \bar{n}, |\sigma| = i, m(\sigma) = m(v)\}$$

and $C_i(I) = \cup_{t=0}^{s} C_{it}(I)$. A variant of Theorem 2.1 is the following:

Proposition 2.4. *The elements of $C_i(I)$ are cycles in $K_i(x; S/I)$ and their homology classes form a basis in $H_i(x; S/I)$ for $i \geq 2$, when $n = 2$.*

Since Lemma 2.3 works only in the case $n = 2$ this gives almost nothing more than Theorem 2.1. Unfortunately the ideals of type $T = m^{p^j} m^{[p^j]}$ could be bad, for example when $p = 3$, $j = 1$ and $n = 3$, because then $T = m^6 \setminus \{x_1^2 x_2^2 x_3^2\}$.

Let M be a graded S-module and $\beta_{ij}(M) = \dim_K \operatorname{Tor}_S^i(K, M)_j$ the ij-th graded Betti number of M.

Corollary 2.5. *In the assumption of Theorem 2.1 $\beta_{ij}(S/I)$ does not depend on the characteristic of the field K for all i, j.*

For the proof note that $H_i(x; S/I) \cong \operatorname{Tor}_i^S(K, S/I)$ and so $\beta_{ij}(S/I)$ is the sum of some $|B_{it}(I)|$ which has nothing to do with the characteristic of K (see Theorem 2.1).

Remark 2.6. Note that $\beta_{ij}(S/I)$ does not depend on the characteristic of K when I is stable by [5]. In [9] it is shown that the extremal graded Betti numbers of S/I (see [3]) do not depend on the characteristic of K when I is a Borel type ideal (see [7]). In particular this happens for p-Borel ideals and so we might ask whether all the $\beta_{ij}(S/I)$ do not depend on the characteristic of K in the case of p-Borel ideals. Corollary 2.5 gives a small hope.

From now on let I be the p-Borel ideal generated by the monomial $x_{n-1}^{\gamma} x_n^{\alpha}$ for some integer $\gamma, \alpha \geq 0$, that is $I = \prod_{j=0}^{s} ((m_{n-1}^{[p^j]})^{\gamma_j} (m^{[p^j]})^{\alpha_j})$, where $m_{n-1} = (x_1, \ldots, x_{n-1})$, and γ_j, α_j are defined by the p-adic expansions of γ, α, respectively. The main result of this paper is the following:

Theorem 2.7. *Suppose that $\alpha_j + \gamma_j < p$ for all $0 \leq j \leq s$. Then:*

(1) *$H_i(x; S/I)$ has a monomial cycle basis for all $i \geq 2$, and*

(2) *$\beta_{ij}(S/I)$ does not depend on the characteristic of K for all i, j.*

For the proof we need some preparations. Suppose $\alpha > 0$. Let $r = \max\{j : \alpha_j > 0\}$ and set $J = \prod_{j=0}^{s}(m_{n-1}^{[p^j]})^{\gamma_j}$ and

$$I' = \left(\prod_{j=0}^{r-1}(m^{[p^j]})^{\alpha_j}\right)(m^{[p^r]})^{\alpha_r-1}.$$

Then

Lemma 2.8. $(I : x_n^{p^r}) = JI'$.

Proof. Obviously if L, T are some monomial ideals and v is a monomial then $(LT : v) = (L : v)T + L(T : v)$. Applying this fact we get

$$(I : x_n^{p^r}) = J\sum\prod_{j=0}^{r}\prod_{k=1}^{\alpha_j}(m^{[p^j]} : x_n^{c_{jk}}) = J\sum\prod_{j=0}^{r}\prod_{k=1}^{\alpha_j}(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}}),$$

where the sum is taken over all integers $0 \le c_{jk} \le p^j$ such that $\sum_{j=0}^{r}\sum_{k=1}^{\alpha_j} c_{jk} = p^r$. For each j let $\Lambda_j \subset \{k : 1 \le k \le \alpha_j, c_{jk} > 0\}$ be any subset. Set $u = \sum_{j=0}^{r}\sum_{k\in\Lambda_j}(p^j - c_{jk})$. We claim that $x_n^u \prod_{j=0}^{r}(m_{n-1}^{[p^j]})^{\alpha_j-|\Lambda_j|} \subset I'$. Clearly if our claim holds then $(I : x_n^{p^r}) \subset JI'$, the other inclusion being trivial. Note that the claim holds because $u \ge (\sum_{j=0}^{r}|\Lambda_j|p^j) - p^r$. □

Let a be an integer such that $0 \le a \le \alpha$ and $a = \sum_{j=0}^{r} a_j p^j$, $0 \le a_j < p$, the p-adic expansion of a. Set $\alpha_a = \sum_{j,\alpha_j\ge a_j}(\alpha_j - a_j)p^j$ and $\alpha_{aj} = \alpha_j - a_j$ if $\alpha_j \ge a_j$ and 0 otherwise. Set

$$I_a = J\left(\prod_{j=0}^{r}(m^{[p^j]})^{\alpha_{aj}}\right),$$

where J is defined above. Let $\pi : S \to \bar{S} = K[x_1, \ldots, x_{n-1}]$ be the $\bar{S}$-morphism given by $x_n \to 0$.

Lemma 2.9. *$\pi(I : x_n^a)$ is the p-Borel ideal generated by the monomial $x_{n-1}^{\gamma}x_{n-1}^{\alpha_a}$, that is $\pi(I : x_n^a) = \pi(I_a)$.*

Proof. It is enough to show the above equality for the case $\gamma = 0$. As in the proof of Lemma 2.8 we have

$$(I : x_n^a) = \sum\prod_{j=0}^{r}\prod_{k=1}^{\alpha_j}(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}}),$$

where the sum is taken over all integers $0 \le c_{jk} \le p^j$ such that $\sum_{j=0}^{r}\sum_{k=1}^{\alpha_j} c_{jk} = a$. Note that $\pi(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}})$ is $m_{n-1}^{[p^j]}$ if $c_{jk} < p^j$ and $\bar{S}$ otherwise. It follows that $\prod_{j=0}^{r}\prod_{k=1}^{\alpha_j}\pi(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}}) \subset \pi(I_a)$; the equality holds only when $\min\{a_j, \alpha_j\} = |\{k : c_{jk} = p^j\}|$ for all j, k. Hence $\pi(I : x_n^a) = \pi(I_a)$. □

Let $T \subset S$ be an arbitrary ideal and $\bar{T} = \pi(T)$.

Lemma 2.10. *$H_i(x; \bar{S}/\bar{T}) \cong H_i(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T}) \oplus H_{i-1}(x_1, \ldots, x_{n-1}; (\bar{S}/\bar{T})(-1))$ and in particular $\beta_{ij}^{S}(\bar{S}/\bar{T}) = \beta_{ij}^{\bar{S}}(\bar{S}/\bar{T}) + \beta_{i-1,j-1}^{\bar{S}}(\bar{S}/\bar{T})$, where $\beta_{ij}^{S}(\bar{S}/\bar{T})$ is the i, j-th graded Betti number of $\bar{S}/\bar{T}$ over S.*

Proof. By [4, Proposition 1.6.21] we have

$$\begin{aligned} H_i(x; \bar{S}/\bar{T}) &\cong H_i(x; S/(\bar{T}, x_n)) \cong H_i(x_1, \ldots, x_{n-1}; S/(\bar{T}S)) \otimes_S (\wedge^1 S) \\ &\cong H_i(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T}) \otimes_S (\wedge^1 S), \end{aligned}$$

and the last isomorphism follows because S is flat over $\bar{S}$.

Because of the above isomorphism we may write

$$H_i(x; \bar{S}/\bar{T}) = H_i(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T}) \oplus H_{i-1}(x_1, \ldots, x_{n-1}; (\bar{S}/\bar{T})(-1)) \wedge e_n.$$

where by abuse of notation we write

$$H_{i-1}(x_1,\dots,x_{n-1};(\bar S/\bar T)(-1))\wedge e_n$$

for $\{\mathrm{cls}(z\wedge e_n) : z \text{ cycle of } K_{i-1}(x_1,\dots,x_{n-1};(\bar S/\bar T)(-1))\}$.

Finally, note that this is enough because $\beta^{\bar S}_{ij}(\bar S/\bar T)=\dim_K \mathrm{Tor}^{\bar S}_i(K,\bar S/\bar T)_j=\dim_K H_i(x_1,\dots,x_{n-1};\bar S/\bar T)_j$ and $\beta^{\bar S}_{i-1,j-1}(\bar S/\bar T)=\beta^{\bar S}_{i-1,j}((\bar S/\bar T)(-1))$. $\square$

We have the following multigraded exact sequence:

$$0\to S/(T:x_n)(-1)\to S/T\to \bar S/\bar T\to 0, \tag{$*$}$$

where the first map is given by multiplication with x_n. Applying the Koszul homology long exact sequence to $(*)$ we get the following multigraded exact sequence:

$$H_i(x;S/(T:x_n)(-1))\to H_i(x;S/T)\to H_i(x;\bar S/\bar T)\to H_{i-1}(x;S/(T:x_n)(-1)), \tag{$**$}$$

where we denote by δ_i the last map above. The next lemma describes how δ_i acts.

Lemma 2.11. *δ_i maps $H_i(x_1,\dots,x_{n-1};\bar S/\bar T)$ to zero and if z is a cycle of $K_{i-1}(x_1,\dots,x_{n-1};\bar S/\bar T)$ then δ_i maps $\mathrm{cls}(z\wedge e_n)$ to*

$$(-1)^{i-1}\mathrm{cls}(z)\in H_{i-1}(x_1,\dots,x_{n-1};S/(T:x_n)(-1)).$$

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & K_i(x;S/(T:x_n)(-1)) & \to & K_i(x;S/T) & \to & K_i(x;\bar S/\bar T) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_{i-1}(x;S/(T:x_n)(-1)) & \to & K_{i-1}(x;S/T) & \to & K_{i-1}(x;\bar S/\bar T) & \to & 0
\end{array}$$

where the vertical maps are given by ∂. Let w be a cycle of $K_i(x;\bar S/\bar T)$. By construction of δ_i we must lift w to an element $v\in K_i(x;S/T)$. Then $\partial(v)=x_n y$ for a cycle $y\in K_{i-1}(x;S/(T:x_n)(-1))$ and we may write $\delta_i(\mathrm{cls}(w))=\mathrm{cls}(y)$. Here we may take $v=w\in K_i(x;S/T)$ which is a cycle. Then we have $y=0$ and so $\delta_i(w)=0$. Now we take $w=z\wedge e_n$ with z a cycle from $K_i(x_1,\dots,x_{n-1};\bar S/\bar T)$. As in the first case we may take $v=z\wedge e_n$ but this time this is not a cycle in $K_i(x;S/T)$. We have $\partial(z\wedge e_n)=\partial(z)\wedge e_n+(-1)^{i-1}x_nz=(-1)^{i-1}x_nz$ since $\partial(z)=0$. Then $\delta_i(\mathrm{cls}(z\wedge e_n))=(-1)^{i-1}\mathrm{cls}(z)$. $\square$

Let f_i be the composite map $H_i(x;S/T)\to H_i(x;\bar S/\bar T)\xrightarrow{q_1}H_i(x_1,\dots,x_{n-1};\bar S/\bar T)$ where q_1 is the first projection of the direct sum given by Lemma 2.10. Then f_i has a canonical section ρ_i^T given by $\mathrm{cls}(z)\to\mathrm{cls}(z)\in H_i(x;S/T)$, z being a cycle of $K_i(x_1,\dots,x_{n-1};\bar S/\bar T)$. Let $\eta_i^T : H_i(x_1,\dots,x_{n-1};(\bar S/\bar T)(-1))\to H_i(x;(\bar S/\pi(T:x_n)(-1)))$ be the canonical map associated with the surjection $\bar S/\bar T\to\bar S/\pi(T:x_n)$ and q_2 be the second projection of the direct sum given by Lemma 2.10.

Corollary 2.12. *The following statements hold:*

(1) $\delta_{i+1}=(-1)^i\rho_i^{(T:x_n)}\eta_i^Tq_2$,
(2) $\mathrm{Ker}\,\delta_{i+1}\cong H_i(x_1,\dots,x_{n-1};\bar S/\bar T)\oplus\mathrm{Ker}\,\eta_i^T$,
(3) $\mathrm{Im}\,\delta_{i+1}\cong\mathrm{Im}\,\eta_i^T$.

Lemma 2.13. *Let ue_σ, $u\in S$ monomial with $m(u)<n$. Suppose that ue_σ is a monomial cycle of $K_i(x_1,\dots,x_{n-1};(\bar S/\bar T)(-1))$ for a monomial ideal $T\subset S$ and $\eta_i^T(ue_\sigma)=0$ in $K_i(x_1,\dots,x_{n-1};(\bar S/\pi(T:x_n))(-1))$. Then $ue_\sigma\wedge e_n$ is a cycle in $K_{i+1}(x;S/T)$ inducing a lifting of $\mathrm{cls}(ue_\sigma)$ by the composite map*

$$\begin{aligned}H_{i+1}(x;S/T)&\to H_{i+1}(x;\bar S/\bar T)\xrightarrow{q_2}H_i(x_1,\dots,x_{n-1};(\bar S/\bar T)(-1))\wedge e_n\\ &\cong H_i(x_1,\dots,x_{n-1};(\bar S/\bar T)(-1)).\end{aligned}$$

Proof. We have $x_j u \in \bar{T} \subset T$ for $j \in \sigma$ because ue_σ is in $K_i(x_1, \ldots, x_{n-1}; (\bar{S}/\bar{T})(-1))$ a monomial cycle. Since $\eta_i^T(ue_\sigma) = 0$ in $K_i(x_1, \ldots, x_{n-1}; (\bar{S}/\pi(T : x_n))(-1))$ we get $u \in \pi(T : x_n) \subset (T : x_n)$. Thus $x_n u \in T$ and so $ue_\sigma \wedge e_n$ is a monomial cycle in $K_{i+1}(x; S/T)$. $\square$

Now we may return to give the proof of Theorem 2.7.

Proof. Apply induction on $c = \sum_{j=0}^{s} \alpha_j p^j$. If $c = 0$ then we are in the case of Theorem 2.1 and Corollary 2.5. Suppose $c > 0$ and set $r = \max\{j : \alpha_j \neq 0\}$. By Lemma 2.8 $I' = (I : x_n^{p^r})$ is the p-Borel ideal generated by the monomial $x_{n-1}^{\gamma} x_n^{\alpha - p^r}$ and from the induction hypothesis $H_i(x; S/I')$ has a monomial cyclic basis and $\beta_{ij}^S(S/I')$ does not depend on the characteristic of K for all i, j.

Let $0 \le a \le p^r$ be an integer. By decreasing induction we show that $H_i(x; S/(I : x_n^a))$ has a monomial cycle basis and $\beta_{ij}^S(S/(I : x_n^a))$ does not depend on the characteristic of K for all i, j. Above we saw the case $a = p^r$. Suppose $a < p^r$. The exact multigraded sequence $(**)$ given before Lemma 2.11 with Corollary 2.12 gives for $T = (I : x_n^a)$ the following exact multigraded sequence:

$$\begin{aligned} 0 \to \operatorname{Im}\delta_{i+1} &\cong \operatorname{Im}\eta_i^{(I:x_n^a)} \to H_i(x; (S/(I : x_n^{a+1}))(-1)) \to H_i(x; S/(I : x_n^a)) \to \operatorname{Ker}\delta_i \\ &\cong H_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(I : x_n^a)) \oplus \operatorname{Ker}\eta_{i-1}^{(I:x_n^a)} \to 0, \end{aligned}$$

where $\eta_i^{(I:x_n^a)} : H_i(x_1, \ldots, x_{n-1}; (\bar{S}/\pi(I : x_n^a))(-1)) \to H_i(x_1, \ldots, x_{n-1}; (\bar{S}/\pi(I : x_n^{a+1}))(-1))$ is given before Corollary 2.12. By Lemma 2.9 we see that $\pi(I : x_n^a) = \pi(I_a)$ is the p-Borel ideal generated by a power of x_{n-1} and it is subject to Theorem 2.1 and Corollary 2.5 because $\gamma_j + \alpha_{aj} \le \gamma_j + \alpha_j < p$ for all j. In particular, $B_i(\pi(I : x_n^a)) = B_i(\pi(I_a))$ is in $H_i(x_1, \ldots, x_{n-1}; (\bar{S}/\pi(I : x_n^a))(-1))$ a monomial cycle basis. Also note that $\operatorname{Im}\eta_i^{(I:x_n^a)}$ is generated by classes of monomial cycles and so $\operatorname{Im}\delta_{i+1}$ is also generated by classes of monomial cycles since $\rho_i^{(I:x_n^{a+1})}$ preserves such classes. Thus we may choose a monomial cycle basis in $\operatorname{Im}\delta_{i+1}$.

Using the induction hypothesis on a we see that $H_i(x; S/(I : x_n^{a+1}))$ has a monomial cyclic basis and $\beta_{ij}^S(S/(I : x_n^{a+1}))$ does not depend on the characteristic of K for all i, j. Then the conclusion follows from the above multigraded exact sequence if we show the following statements:

(1) A monomial cycle basis of $H_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(I : x_n^a))$ can be lifted to a monomial cycle subset of $K_i(x; S/(I : x_n^a))$.
(2) $\operatorname{Ker}\eta_i^{(I:x_n^a)}$ has a monomial cycle basis which can be lifted to a monomial cycle subset of $K_i(x; S/(I : x_n^a))$.
(3) $\dim_K \operatorname{Ker}\eta_i^{(I:x_n^a)}$ does not depend on the characteristic of K.

Actually (1) follows because a cycle from $K_i(x_1, \ldots, x_{n-1}; (\bar{S}/\pi(I : x_n^a))(-1))$ can be viewed as a cycle from $K_i(x_1, \ldots, x_{n-1}; (S/(I : x_n^a))(-1))$ as was already seen in the proof of Lemma 2.11. For (2) and (3) we study how $\eta_i^{\pi(I_a)}$ acts on $B_i(\pi(I_a))$. We have the following cases:

Case $a_0 < p - 1$.

Then $a + 1 = (\sum_{j>0} a_j p^j) + (a_0 + 1)$ is the p-adic expansion of $a + 1$, that is $(a + 1)_j = a_j$ for all $j > 0$ and $(a + 1)_0 = a_0 + 1$. We have $\alpha_{a+1,j} = \alpha_{aj}$ for $j > 0$. If $\alpha_0 \le a_0$ then $\alpha_{a+1,0} = 0$ and $\eta_i^{\pi(I_a)}$ acts identically because $\pi(I_a) = \pi(I_{a+1})$. Thus $\operatorname{Ker}\eta_i^{\pi(I_a)} = 0$. If $\alpha_0 > a_0$ then $\alpha_{a+1,0} = \alpha_{a,0} - 1$. Thus $\eta_i^{\pi(I_a)}$ acts identically on $\cup_{t \ge 1} B_{it}(\pi(I_a))$ and sends $B_{io}(\pi(I_a))$ in zero since if $v \in G(m_{n-1}^{\alpha_{a0}})$ then $v' \in G(m_{n-1}^{\alpha_{a0}-1})$. So the monomial cyclic basis of $\operatorname{Ker}\eta_i^{\pi(I_a)}$ is given by $B_{io}(\pi(I_a))$.

Case $a_j = p - 1$ for $0 \le j < t$, $a_t < p - 1$.

Then $a + 1 = (a_t + 1)p^t + \sum_{j>t} a_j p^j$ is the p-adic expansion of $a + 1$, that is $(a + 1)_j = 0$ for $j < t$, $(a + 1)_j = a_t + 1$ for $j = t$ and $(a + 1)_j = a_j$ for $j > t$. We have $\alpha_{a+1,j} = \alpha_{a,j}$ for $j > t$ and so $\eta_i^{\pi(I_a)}$ acts identically on $\cup_{j>t} B_{ij}(\pi(I_a))$. If $\alpha_t \le a_t$ then $\alpha_{at} = \alpha_{a+1,t}$ and $\eta_i^{\pi(I_a)}$ acts identically on $B_{it}(\pi(I_a))$. If $\alpha_t > a_t$ then $\alpha_{a+1,t} = \alpha_{at} - 1$ and $\eta_i^{\pi(I_a)}$ sends $B_{it}(\pi(I_a))$ to zero. Suppose $j < t$. Then $\alpha_{a,j} = 0$ and so $B_{ij}(\pi(I_a)) = \emptyset$.

Consequently, given $j \ge 0$, in both cases $\eta_i^{\pi(I_a)}$ either acts identically on $B_{ij}(\pi(I_a))$ or sends it to zero. It follows that $\operatorname{Ker}\eta_i^{\pi(I_a)}$ has a monomial cyclic basis which can be lifted to $H_i(x; S/(I : x_n^a))$ by Lemma 2.13. It consists of some $B_{ij}(\pi(I_a))$ whose cardinal does not depend on the characteristic of K. This ends our decreasing induction. Thus the ideal $(I : x_n^a)$ satisfies the conditions (1), (2) from Theorem 2.7 for all $0 \le a \le p^r$. In particular this holds for $a = 0$. $\square$

Remark 2.14. Note that the above proof shows also that some non-principal p-Borel ideals of the form $(I : x_n^a)$ have monomial cyclic bases.

We end this section with an example illustrating the proof of Theorem 2.7.

Example 2.15. Let $n = 3$, $p = 2$, $S = K[x_1, x_2, x_3]$, $m = (x_1, x_2, x_3)$, $I = m^{[2]}m$. Using Theorem 2.1, a cyclic basis of $H_2(x; S/I)$ is given by $B_{21}(I) = \{x_1x_2e_1 \wedge e_2, x_1x_3e_1 \wedge e_3, x_2x_3e_2 \wedge e_3\}$ and $B_{20}(I) = \{x_i^2e_1 \wedge e_2, x_i^2e_1 \wedge e_3, x_i^2e_2 \wedge e_3 : 1 \le i \le 3\}$. We will show this independently using the procedure from the proof of Theorem 2.7. Let $\pi : S \to \bar{S} = K[x_1, x_2]$ be the $\bar{S}$-morphism given by $x_3 \to 0$. Then $\bar{I} = \pi(I) = m_2^{[2]}m_2$, where $m_2 = (x_1, x_2)$ and $\pi(I : x_3) = m_2^{[2]}$, $(I : x_3^2) = m$. Note that monomial cyclic bases of $H_2(x_1, x_2; \bar{S}/m_2^{[2]}m_2)$, $H_2(x_1, x_2; \bar{S}/m_2^{[2]})$ are given by $B_{21}(\bar{I}) = \{x_1x_2e_1 \wedge e_2\} = B_{21}(\pi(I : x_3))$ and $B_{20}(\bar{I}) = \{x_1^2e_1 \wedge e_2, x_2^2e_1 \wedge e_2\}$. The map η_2^I maps $B_{20}(\bar{I})$ to zero and it is the identity on $B_{21}(\bar{I})$. The maps $\eta_1^I, \eta_i^{(I:x_3)}$, $i = 1, 2$, are zero maps.

We have the following multigraded exact sequence:

$$\begin{aligned}\operatorname{Im}\eta_2^{(I:x_3)} = {} & 0 \to H_2(x; (S/m)(-1)) \to H_2(x; S/(I : x_3)) \\ & \to H_2(x_1, x_2; \bar{S}/m_2^{[2]}) \oplus H_1(x_1, x_2; (\bar{S}/m_2^{[2]})(-1)) \wedge e_3 \to 0.\end{aligned}$$

As the monomial cyclic bases of $H_2(x; S/m)$, $H_2(x_1, x_2; \bar{S}/m_2^{[2]})$, $H_1(x_1, x_2; \bar{S}/m_2^{[2]})$ are $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$, $\{x_1x_2e_1 \wedge e_2\}$, $\{x_1e_1, x_2e_2\}$, respectively, we see that a monomial cycle basis in $H_2(x; S/(I : x_3))$ is given by

$$T_2(I : x_3) = \{x_3e_1 \wedge e_2, x_3e_1 \wedge e_3, x_3e_2 \wedge e_3, x_1x_2e_1 \wedge e_2, x_1e_1 \wedge e_3, x_2e_2 \wedge e_3\}.$$

Now consider the multigraded exact sequence

$$\begin{aligned}& 0 \to \operatorname{Im}\eta_2^I \to H_2(x; (S/(I : x_3))(-1)) \to H_2(x; S/I) \\ & \to H_2(x_1, x_2; \bar{S}/m_2^{[2]}m_2) \oplus H_1(x_1, x_2; (\bar{S}/m_2^{[2]}m_2)(-1)) \wedge e_3 \to 0.\end{aligned}$$

As the monomial cyclic bases of $\operatorname{Im}\eta_2^I$, $H_2(x_1, x_2; \bar{S}/m_2^{[2]}m_2)$, $H_1(x_1, x_2; \bar{S}/m_2^{[2]}m_2)$ are $\{x_1x_2e_1 \wedge e_2\}$, $T_2(m_2^{[2]}m_2) = \{x_1x_2e_1 \wedge e_2, x_1^2e_1 \wedge e_2, x_2^2e_1 \wedge e_2\}$, $T_1(m_2^{[2]}m_2) = \{x_1^2e_1, x_2^2e_1, x_1^2e_2, x_2^2e_2\}$, respectively, we see that a monomial cycle basis in $H_2(x; S/I)$ is given by

$$x_3[T_2(I : x_3) \setminus \{x_1x_2e_1 \wedge e_2\}] \cup T_2(m_2^{[2]}m_2) \cup T_1(m_2^{[2]}m_2) = B_{20}(I) \cup B_{21}(I) = B_2(I).$$

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